Partial derivatives and differentials

The definition of partial differential and differentials is a little harder, we will not deal with it here and you, of course, learn it in way your professor requires. We will try to teach you how to solv some types of tasks…

First a few words about labeling: usually we have function $z = z(x, y)$, so:

$\frac{\partial z}{\partial x} \rightarrow$ stands for partial derivative " by x"

$\frac{\partial z}{\partial y} \rightarrow$ stands for partial derivative " by y"

$\frac{\partial^2 z}{\partial x^2} \rightarrow$ stands for double partial derivative "by x" and is calculated $\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$

$\frac{\partial^2 z}{\partial y^2} \rightarrow$ stands for double partial derivative "by y" and is calculated $\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right)$

$\frac{\partial^2 z}{\partial y \partial x}$ and $\frac{\partial^2 z}{\partial x \partial y}$ are the mixed partial derivatives : $\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)$ and $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)$

Here is the **most important** thing to remember the following:

$\rightarrow$ **When looking for a partial derivative "by *x*" then treat *y* as a constant (number)**

$\rightarrow$ **When looking for a partial derivative "by *y*" then treat *x* as a constant (number)**

Example 1.

Determine the first partial derivatives of function $z = x^2 + y^2 - 2x + 3y$

Solution:

First look $\frac{\partial z}{\partial x} \rightarrow$ partial derivative " by x". This means that y is a constant. $z = x^2 + \boxed{y^2} - 2x + \boxed{3y}$

We know that the derivative of the constant is 0 when not connected to a function, so :

$z = x^2 + y^2 - 2x + 3y$

$\frac{\partial z}{\partial x} = 2x + 0 - 2 + 0 = \boxed{2x - 2}$

Now look for partial derivative " by y":

$z = \boxed{x^2} + y^2 - \boxed{2x} + 3y$ constants are rounded, and the derivative of them is 0

$$z = x^2 + y^2 - 2x + 3y$$

$$\frac{\partial z}{\partial y} = 0 + 2y - 0 + 3 = \boxed{2y + 3}$$

Example 2.

Determine the first partial derivatives of function $z = 3x^3 y - 6xy + 5y^2 + 7x - 12y$

Solution:

$$z = 3x^3 y - 6xy + 5y^2 + 7x - 12y$$

$$\frac{\partial z}{\partial x} = 3y \cdot 3x^2 - 6y \cdot 1 + 0 + 7 - 0 = \boxed{9x^2 y - 6y + 7}$$

Now, constants are related to the function, rewrite them, and look normal derivative of a function "by x", for

Example: for $3x^3 y$ constant is 3y and derivate from x^3 is $3x^2$.

To find the first partial derivative "by y":

$$z = 3x^3 y - 6xy + 5y^2 + 7x - 12y$$

$$\frac{\partial z}{\partial y} = 3x^3 \cdot 1 - 6x \cdot 1 + 10y + 0 - 12 = \boxed{3x^3 - 6x + 10y - 12}$$

Example 3.

Determine the first partial derivatives of function $z = \frac{3x}{y} + \frac{7y}{x}$

Solution:

$$z = \frac{3x}{y} + \frac{7y}{x}$$

$$\frac{\partial z}{\partial x} = \frac{3}{y} \cdot 1 + 7y \cdot (-\frac{1}{x^2}) = \boxed{\frac{3}{y} - \frac{7y}{x^2}}$$

$$\frac{\partial z}{\partial y} = 3x \cdot (-\frac{1}{y^2}) + \frac{7}{x} \cdot 1 = \boxed{-\frac{3x}{y^2} + \frac{7}{x}}$$

Example 4.

Determine the first partial derivatives of function $u = \ln(x + y^2)$

Solution:

Watch out, here we have a complex functions derivative:

$$u = \ln(x + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{1}{x + y^2} \cdot (x + y^2)`_{\text{po x}} = \frac{1}{x + y^2} \cdot (1 + 0) = \boxed{\frac{1}{x + y^2}}$$

$$\frac{\partial u}{\partial y} = \frac{1}{x + y^2} \cdot (x + y^2)`_{\text{po y}} = \frac{1}{x + y^2} \cdot (0 + 2y) = \boxed{\frac{2y}{x + y^2}}$$

Example 5.

Determine the first partial derivatives of function $z = x^y$

Solution:

$$z = x^y$$

$\frac{\partial z}{\partial x} = y \cdot x^{y-1}$ Here we work as $(x^{\Theta})` = \Theta \cdot x^{\Theta - 1}$

$$\frac{\partial z}{\partial y} = x^y \cdot \ln x$$

For the partial derivative “by y” work as: $(a^y)` = a^y \ln a$

Example 6.

Determine the first partial derivatives of function $z=\frac{x+y}{x^2+y^2}$

Solution:

Here we work as a $(\frac{u}{v})`=\frac{u`v-v`u}{v^2}$

$$z=\frac{x+y}{x^2+y^2}$$

$$\frac{\partial z}{\partial x}=\frac{(x+y)`_{\text{po x}}\cdot(x^2+y^2)-(x^2+y^2)`_{\text{po x}}\cdot(x+y)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial x}=\frac{1\cdot(x^2+y^2)-2x\cdot(x+y)}{(x^2+y^2)^2}=\frac{x^2+y^2-2x^2-2xy}{(x^2+y^2)^2}=\boxed{\frac{-x^2+y^2-2xy}{(x^2+y^2)^2}}$$

$$\frac{\partial z}{\partial y}=\frac{(x+y)`_{\text{po y}}\cdot(x^2+y^2)-(x^2+y^2)`_{\text{po y}}\cdot(x+y)}{(x^2+y^2)^2}$$

$$\frac{\partial z}{\partial y}=\frac{1\cdot(x^2+y^2)-2y\cdot(x+y)}{(x^2+y^2)^2}=\frac{x^2+y^2-2xy-2y^2}{(x^2+y^2)^2}=\boxed{\frac{x^2-y^2-2xy}{(x^2+y^2)^2}}$$

Example 7.

Determine the first partial derivatives of function $u=\ln(x^2+y^2+z^2)$

Solution:

$$u=\ln(x^2+y^2+z^2)$$

$$\frac{\partial u}{\partial x}=\frac{1}{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)`_{\text{by x}}=\frac{1}{x^2+y^2+z^2}\cdot 2x=\boxed{\frac{2x}{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial y}=\frac{1}{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)`_{\text{by y}}=\frac{1}{x^2+y^2+z^2}\cdot 2y=\boxed{\frac{2y}{x^2+y^2+z^2}}$$

$$\frac{\partial u}{\partial z}=\frac{1}{x^2+y^2+z^2}\cdot(x^2+y^2+z^2)`_{\text{by z}}=\frac{1}{x^2+y^2+z^2}\cdot 2z=\boxed{\frac{2z}{x^2+y^2+z^2}}$$

Example 8.

Determine the first partial derivatives of function $u = \left(\frac{x}{y}\right)^z$

Solution:

$$u = \left(\frac{x}{y}\right)^z$$

$$\frac{\partial u}{\partial x} = z\left(\frac{x}{y}\right)^{z-1} \cdot \left(\frac{x}{y}\right)^{`}_{\text{by x}} = z\left(\frac{x}{y}\right)^{z-1} \cdot \frac{1}{y} = \boxed{\frac{z}{y} \cdot \left(\frac{x}{y}\right)^{z-1}}$$

$$\frac{\partial u}{\partial y} = z\left(\frac{x}{y}\right)^{z-1} \cdot \left(\frac{x}{y}\right)^{`}_{\text{by y}} = z\left(\frac{x}{y}\right)^{z-1} \cdot x \cdot \left(-\frac{1}{y^2}\right) = \boxed{-\frac{z}{y} \cdot \left(\frac{x}{y}\right)^{z}}$$

$$\frac{\partial u}{\partial z} = \boxed{\left(\frac{x}{y}\right)^z \cdot \ln\left(\frac{x}{y}\right)}$$

Example 9.

Find $\frac{\partial^2 z}{\partial x^2} = ?,\quad \frac{\partial^2 z}{\partial x \partial y} = ?,\quad \frac{\partial^2 z}{\partial y \partial x} = ?,\quad \frac{\partial^2 z}{\partial y^2} = ?$ **for function** $z = 3x^3y - 6xy + 5y^2 + 7x - 12y$

Solution:

Of course, first we find the first partial derivatives:

$$z = 3x^3y - 6xy + 5y^2 + 7x - 12y$$

$$\frac{\partial z}{\partial x} = 3y \cdot 3x^2 - 6y \cdot 1 + 0 + 7 - 0 = \boxed{9x^2y - 6y + 7}$$

$$\frac{\partial z}{\partial y} = 3x^3 \cdot 1 - 6x \cdot 1 + 10y + 0 - 12 = \boxed{3x^3 - 6x + 10y - 12}$$

Now using them look further:

$$\frac{\partial z}{\partial x} = \boxed{9x^2y - 6y + 7}$$

$$\frac{\partial z}{\partial y} = \boxed{3x^3 - 6x + 10y - 12}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(9x^2y - 6y + 7) = \boxed{18xy}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(9x^2y - 6y + 7) = \boxed{9x^2 - 6}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(3x^3 - 6x + 10y - 12) = \boxed{9x^2 - 6}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(3x^3 - 6x + 10y - 12) = \boxed{10}$$

Example 10.

Find $\frac{\partial^2 z}{\partial x^2} = ?, \quad \frac{\partial^2 z}{\partial x \partial y} = ?, \quad \frac{\partial^2 z}{\partial y \partial x} = ?, \quad \frac{\partial^2 z}{\partial y^2} = ?$ **for function** $z = e^{xy}$

<u>Solution:</u>

$$z = e^{xy}$$

$$\frac{\partial z}{\partial x} = e^{xy} \cdot (xy)`_{\text{by x}} = e^{xy} \cdot y = ye^{xy} \rightarrow \boxed{\frac{\partial z}{\partial x} = ye^{xy}}$$

$$\frac{\partial z}{\partial y} = e^{xy} \cdot (xy)`_{\text{by y}} = e^{xy} \cdot x = xe^{xy} \rightarrow \boxed{\frac{\partial z}{\partial y} = xe^{xy}}$$

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(ye^{xy}) = y \cdot ye^{xy} = \boxed{y^2e^{xy}}$$

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}(ye^{xy}) = 1 \cdot e^{xy} + xe^{xy} \cdot y = \boxed{e^{xy}(1 + xy)}$$

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}(xe^{xy}) = 1 \cdot e^{xy} + ye^{xy} \cdot x = \boxed{e^{xy}(1 + xy)}$$

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial y}(xe^{xy}) = x \cdot xe^{xy} = \boxed{x^2e^{xy}}$$

Example 11.

Show that $\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=2z$ **if** $z=x^y$

Solution:

In this type of task, first we find the partial derivatives that occur in the task (here on the left side of the equality)

Replace them and arrange to get the right side:

$$z=x^y$$

$$\frac{\partial z}{\partial x}=y\cdot x^{y-1}$$

$$\frac{\partial z}{\partial y}=x^y\cdot\ln x$$

Now rewrite the left side and replace partial derivatives:

$$\frac{x}{y}\frac{\partial z}{\partial x}+\frac{1}{\ln x}\frac{\partial z}{\partial y}=$$

$$\frac{x}{\cancel{y}}\cdot\cancel{y}x^{y-1}+\frac{1}{\cancel{\ln x}}\cdot x^y\cancel{\ln x}=$$

$$x^y+x^y=2x^y=2z$$

We prove the required equality!

Example 12.

Show that $x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=x$ **if** $z=x+\varphi(xy)$

Solution:

$$z=x+\varphi(xy)$$

$$\frac{\partial z}{\partial x}=1+\varphi`(xy)\cdot(xy)`_{\text{by x}}=1+\varphi`(xy)\cdot y=\boxed{1+y\cdot\varphi`(xy)}$$

$$\frac{\partial z}{\partial y}=0+\varphi`(xy)\cdot(xy)`_{\text{by y}}=\varphi`(xy)\cdot x=\boxed{x\cdot\varphi`(xy)}$$

replace this in:

$$x\frac{\partial z}{\partial x}-y\frac{\partial z}{\partial y}=$$

$$x\cdot(1+y\cdot\varphi`(xy))-y\cdot x\cdot\varphi`(xy)=$$

$$x+\cancel{xy\varphi`(xy)}-\cancel{xy\varphi`(xy)}=\boxed{x}$$

We got the right side of equality.

Example 13.

Show that $y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=0$ **if** $z=\varphi(\sqrt{x^2+y^2})$

Solution:

$$z=\varphi(\sqrt{x^2+y^2})$$

$$\frac{\partial z}{\partial x}=\varphi`(\sqrt{x^2+y^2})\cdot(\sqrt{x^2+y^2})`_{\text{by x}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{2\sqrt{x^2+y^2}}(x^2+y^2)`_{\text{by x}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}x$$

$$\boxed{\frac{\partial z}{\partial x}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{x}{\sqrt{x^2+y^2}}}$$

$$\frac{\partial z}{\partial y}=\varphi`(\sqrt{x^2+y^2})\cdot(\sqrt{x^2+y^2})`_{\text{by y}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{2\sqrt{x^2+y^2}}(x^2+y^2)`_{\text{by y}}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{1}{\cancel{2}\sqrt{x^2+y^2}}\cdot\cancel{2}y$$

$$\boxed{\frac{\partial z}{\partial y}=\varphi`(\sqrt{x^2+y^2})\cdot\frac{y}{\sqrt{x^2+y^2}}}$$

$$y\frac{\partial z}{\partial x}-x\frac{\partial z}{\partial y}=$$

$$y\cdot\varphi`(\sqrt{x^2+y^2})\cdot\frac{x}{\sqrt{x^2+y^2}}-x\cdot\varphi`(\sqrt{x^2+y^2})\cdot\frac{x}{\sqrt{x^2+y^2}}=$$

$$\cancel{\varphi`(\sqrt{x^2+y^2})\cdot\frac{xy}{\sqrt{x^2+y^2}}}-\cancel{\varphi`(\sqrt{x^2+y^2})\cdot\frac{xy}{\sqrt{x^2+y^2}}}=0$$

TOTAL Differential for function $z = z(x, y)$ mark as dz is required by the formula:

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

Thus, we find partial derivatives and replace them in the formula…

Example 14.

Find the total differential of the following functions:

a) $z = x^2 y$

b) $u = \frac{z}{x^2 + y^2}$

Solution:

a)

$$z = x^2 y$$
$$\frac{\partial z}{\partial x} = 2xy$$
$$\frac{\partial z}{\partial y} = x^2$$
$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$
$$\boxed{dz = 2xydx + x^2 dy}$$

b)

$$u = \frac{z}{x^2 + y^2}$$
$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy + \frac{\partial u}{\partial z} dz$$
$$\frac{\partial u}{\partial x} = z \cdot (-\frac{1}{(x^2 + y^2)^2}) \cdot 2x = -\frac{2xz}{(x^2 + y^2)^2} \rightarrow \boxed{\frac{\partial u}{\partial x} = -\frac{2xz}{(x^2 + y^2)^2}}$$
$$\frac{\partial u}{\partial y} = z \cdot (-\frac{1}{(x^2 + y^2)^2}) \cdot 2y = -\frac{2yz}{(x^2 + y^2)^2} \rightarrow \boxed{\frac{\partial u}{\partial y} = -\frac{2yz}{(x^2 + y^2)^2}}$$
$$\boxed{\frac{\partial u}{\partial z} = \frac{1}{x^2 + y^2}}$$

Now replace in this formula:

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

$$\boxed{du = -\frac{2xz}{(x^2+y^2)^2}dx - \frac{2yz}{(x^2+y^2)^2}dy + \frac{1}{x^2+y^2}dz}$$

If the tasks require higher-order total differential, then we do, for example:

For $u = u(x, y)$ is $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$ and total differentials of the second and third row would be:

$$d^2u = (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy)^2 = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2$$

$$d^3u = (\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy)^3 = \frac{\partial^3 u}{\partial x^3}dx^3 + 3\frac{\partial^3 u}{\partial x^2 \partial y}dx^2dy + 3\frac{\partial^3 u}{\partial x \partial y^2}dxdy^2 + \frac{\partial^3 u}{\partial y^3}dy^3$$

Example 15.

If $u = x^3 + y^3 - 3x^2y + 3xy^2$ **, find** d^2u

Solution:

$$u = x^3 + y^3 - 3x^2y + 3xy^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 6xy + 3y^2 \to \frac{\partial^2 u}{\partial x^2} = 6x - 6y$$

$$\frac{\partial u}{\partial y} = 3y^2 - 3x^2 + 6xy \to \frac{\partial^2 u}{\partial y^2} = 6y + 6x$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x} = -6x + 6y$$

$$d^2u = \frac{\partial^2 u}{\partial x^2}dx^2 + 2\frac{\partial^2 u}{\partial x \partial y}dxdy + \frac{\partial^2 u}{\partial y^2}dy^2$$

$$d^2u = (6x - 6y)dx^2 + 2(-6x + 6y)dxdy + (6y + 6x)dy^2$$